Abstract
In this work, we presented a nonlinear time fractional model of measles in order to understand the outbreaks of this epidemic disease. The stability analysis and sensitivity analysis of the model is provided and the certain threshold value of the basic reproduction number $R_0 > 1$, disease free and endemic equilibrium point of the model is also calculated. We develop unconditionally convergent nonstandard finite difference scheme by applying Mickens approach $O(h)=h+O(h^2)$. This method proved to be very efficient technique for solving epidemic models to control the infection diseases. We also discussed the qualitative behavior of the model and numerical simulations are carried out to support the analytic results.

Keywords: Qualitative Analysis; Stability Analysis; Nonstandard Finite Difference Scheme

Introduction
In human population epidemiology studies play an important role to understand the disease in human population. Often the work of mathematical epidemiology consists of model building, estimation of parameters and investigation of the sensitivity of models to change in the parameters and numerical simulations. Epidemiologists use mathematical models to understand previous outbreaks of diseases and to better understand the dynamics of how infections spread through populations [1]. The research of this kind helps to understand the ratio of disease spread in the population and to control their parameters [2, 3]. These types of diseased models are often called infectious diseases (i.e. the disease which transferred from one person to another person). Measles, rubella, chicken pox, mumps, aids and gonorrhea syphilis are the examples of infectious disease [4, 5]. Rubella virus is highly infectious illness which is also known as morbilli or measles. The virus can be found in the mucus of the throat, nose of an infected adult and child. Measles symptoms caused by Rubeola virus always included fever, coryza (runny nose), conjunctivitis and at least one of the three Cs-coughs. Symptoms appear after the initial infection about 9-11 days [6]. Complications of measles are fairly common but the patients have weak immune system are more likely to be worse such as those with HIV/AIDS or leukemia and those with vitamin deficiency. Healthy children over the age of 5 are less likely to have complications than adults over the age of 20. It is the first and worst eruptive fever occurs during childhood [7, 8].

The NSFD schemes preserve main properties of the differential counterparts, such as positivity, monotonicity, periodicity, stability, and some other invariant including energy and geometrical shapes. It should be emphasized that NSFD schemes can preserve all properties of continuous models for any discretization parameters. The discrete models with these properties are called dynamically consistent [9]. For the last two decades NSFD methods have attracted attention for many researchers and achieved significant results [10]. The property of stability of the set of equilibria of differential equation is one of these results because it plays the essential role in the study of asymptotical behavior of the solutions of the differential equations [11-13]. The construction of differences schemes, which preserve the stability of the equilibrium points, is important in numerical simulation of differential equations. There are many works concerning the elementary stable schemes. The typical results are for general dynamical systems and for other specific systems [14, 15].

Mathematical Model
W.O Kermack and A.G.MecKendric are those persons who are inventor of the diseases models, and they played an important role in Mathematical epidemiology. In perposed model population is divided into three groups which denoted by S (susceptible), I (infected) and R (recovered). The class S of susceptible is increased by birth at a rate $\pi N$ where both rate $\pi$ and death rate $\mu$ are same, so susceptible is increased by $\mu N$. It is decreased by infection following contacts with infected individuals I at rate $\alpha$. This class is decreased by recovery from infection at a rate $\gamma$ and dimensioned by natural death rate $\mu$. The class I of individuals who have complete protection against disease [16]. The flow chart of the model is representing in (figure 1).

![Figure 1: flow chart of model](image-url)
Following are the equations of the system:

\[
\begin{align*}
\frac{dS}{dt} &= \mu N - \alpha SI - \mu S \tag{1} \\
\frac{dI}{dt} &= \alpha SI - (\gamma + \mu)I \tag{2} \\
\frac{dR}{dt} &= \gamma I - \mu R \tag{3}
\end{align*}
\]

Where \(\mu, \alpha\) and \(\gamma\) represents the per capita removal rate, transitivity rate and per capita recovery rate respectively.

Where \(N = S + I + R\)

With initial conditions 
\(S(0) = 990, I(0) = 10, R(0) = 0\)

**Qualitative Analysis of the Model**

For equilibrium we have,

\[
\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0
\]

This states the population is disease free that is infection dies out of the population and is such that

\[
0 = \mu N - \alpha SI - \mu S
\]

\[
0 = \alpha SI - (\gamma + \mu)I
\]

\[
0 = \gamma I - \mu R
\]

By simplifying the above equations we get disease free equilibrium denoted by \(E_0\) \(i.e E_0 = (N, 0, 0)\) and the endemic equilibrium point denoted by \(E_1\) \(i.e E_1 = \left(\frac{(\gamma + \mu)}{\alpha}, \frac{\alpha \mu N - \mu (\gamma + \mu)}{\alpha (\gamma + \mu)}, \frac{\alpha \mu N - \mu (\gamma + \mu)}{\alpha (\gamma + \mu)}\right)\)

**Reproductive number**

Consider the jacobian matrix \(J\).

\[
J = \begin{bmatrix}
-(\alpha I + \mu) & -\alpha S & 0 \\
\alpha I & \alpha S - (\gamma + \mu) & 0 \\
0 & \gamma & -\mu
\end{bmatrix}
\]

Since the jacobian matrix is \(J = F - V\) where

\[
F = \begin{bmatrix}
-(\alpha I + \mu) & 0 & 0 \\
\alpha I & 0 & 0 \\
0 & \gamma & 0
\end{bmatrix}
\]

\[
V = \begin{bmatrix}
0 & 0 & 0 \\
\alpha S & 0 & 0 \\
0 & \gamma & 0
\end{bmatrix}
\]

We know that \(K = FV^{-1}\) and using the relations \(|K-\lambda I| = 0\) for the Eigen values, which represents reproductive number \(R_0\) \(i.e R_0 = \frac{\alpha N}{(\gamma + \mu)} > 1\)

**Sensitivity analysis of \(R_0\)**

The sensitivity of \(R_0\) is as follows

\[
\frac{\partial R_0}{\partial \alpha} = \frac{N}{(\gamma + \mu)} > 0
\]

\[
\frac{\partial R_0}{\partial \mu} = \frac{-N \alpha}{(\gamma + \mu)^2} < 0
\]

\[
\frac{\partial R_0}{\partial \gamma} = \frac{-N \alpha}{(\gamma + \mu)^2} < 0
\]

\[
\frac{\partial R_0}{\partial N} = \frac{\alpha}{(\gamma + \mu)} > 0
\]

It can be seen that \(R_0\) is most sensitive to change of its parameter, here \(R_0\) is increasing with \(\alpha, N\) and decreasing with \(\mu, \gamma\). In other words it found that the sensitivity analysis shows that prevention is better than to control the disease.

**Stability at the Endemic Equilibrium**

To examine the local stability of endemic equilibrium \(E_1\) we evaluate the jacobian matrix at

\[
E_1 = \left(\frac{(\gamma + \mu)}{\alpha}, \frac{\alpha \mu N - \mu (\gamma + \mu)}{\alpha (\gamma + \mu)}, \frac{\alpha \mu N - \mu (\gamma + \mu)}{\alpha (\gamma + \mu)}\right)
\]

The Matrix which is called Jacobian is of the form:

\[
J = \begin{bmatrix}
-(\alpha I + \mu) & -\alpha S & 0 \\
\alpha I & \alpha S - (\gamma + \mu) & 0 \\
0 & \gamma & -\mu
\end{bmatrix}
\]

We have the stability result that shows that our model is locally asymptotically stable

**Theorem 1:**

The endemic equilibrium \(E_1\) is locally asymptotically stable for \(R_0 > 1\) otherwise unstable.

Proof:

\(E_1\) is locally asymptotically stable if \(Re(\lambda) < 0\) where \(\lambda\) can be calculated from the relation \(|J - \lambda I| = 0\) \(i.e\)

\[
\begin{bmatrix}
-(\alpha I + \mu + \lambda) & -\alpha S & 0 \\
\alpha I & \alpha S - (\gamma + \mu) - \lambda & 0 \\
0 & \gamma & -(\mu + \lambda)
\end{bmatrix}
= 0
\]

\(i.e (\alpha I + \mu + \lambda)(\alpha S - (\gamma + \mu + \lambda))(\mu + \lambda) + \alpha^2 S[I(\mu + \lambda)] = 0\)

The equation which is given above is called characteristic equation. By substituting the values of endemic equilibrium \(E_1\) we get values of \(\lambda\) as follows

\[
\begin{align*}
\lambda_1 &= -\mu \\
\lambda_2 &= \frac{-\mu \pm \sqrt{\mu(4\gamma + 4\mu - 4\beta N)}}{2} \\
\lambda_3 &= \frac{-\mu \pm \sqrt{\mu(4\gamma + 9\mu - 4\beta N)}}{2}
\end{align*}
\]

Real parts are \(\lambda_1 = -\mu, \lambda_2 = \frac{-\mu}{2}, \lambda_3 = \frac{-\mu}{2}\) if \(\lambda < 0\). Therefore, since all the Eigen values are negative. So \(E_1\) is locally asymptotically stable. This proves the proposition.
Nonstandard finite difference (NSFD) scheme

Nonstandard finite difference (NSFD) methods for the numerical integration of differential equations had their genesis in a paper published in 1989 [17-20]. The basic rules to construct such schemes and their application to specific nonlinear equations appear in a variety of publications [9, 14]. In recent years, NSFD discrete models have been constructed and or tested for a wide range on nonlinear dynamical systems [21]. In this section, we design the NSFD scheme that replicates the dynamics of continuous model (1)-(3). Let $Y_k = (S_k, I_k, R_k)$ denote the approximation of $X(t_k)$ where $t_k = k\Delta t$, with $k \in \mathbb{N}$, $h = \Delta t > 0$ be a step size then

$$S^{k+1} - S^k = \frac{\mu N - \alpha S^k I^k - \beta S^k}{\varnothing} h$$

$$I^{k+1} - I^k = \frac{\alpha N S^k + \beta I^k - (\gamma + \mu) I^k}{\varnothing} h$$

$$R^{k+1} - R^k = \frac{\mu S^k - \mu R^k - \mu I^k}{\varnothing} h$$

Which is the purposed NSFD scheme for the given model, where

$$\varnothing = \varnothing(h) = 1 - \frac{e^{-\alpha h} - e^{-\beta h}}{(\gamma + \mu)}$$

The discrete method (4-6) is indeed an NSFD scheme because it is constructed according to Mickens rules [17] formalized as follows in [16].

**Rule 1**

The standard denominator $h = \varnothing(t)$ of the discrete derivatives is replaced by the complex denominator function in Equation (35) which satisfies the asymptotic relation

$$\varnothing(h) = h + O(h^2)$$

Note that the denominator function $\varnothing$ is expected to better capture the dynamics of the continuous model through the presence of the underlying parameters $\gamma, \mu$. In fact, exact schemes for a wide range of dynamical systems involve such complex denominator functions [17-19].

**Rule 2**

Non-linear terms of the right hand side of equation (1)-(3) are approximated in a non-local way. For instance, we have

$$H(t_k)S(t_k) \equiv I_k S_{k+1} \text{ instead of } H(t_k)S(t_k) \equiv I_k S_k$$

**Analysis of the Scheme**

**Theorem 2:** The NSFD scheme (4)-(18) is a dynamical system on the biological feasible domain $\kappa$ of the continuous model (1)-(3).

**Proof:** First we prove the positivity of the scheme (16)-(18). It is to show that the NSFD scheme (16)-(18) takes the explicit form

$$S^{k+1} = \frac{\mu N + S^k}{1 + \alpha (1 + \mu) + \beta} h$$

$$I^{k+1} = \frac{\alpha (S^k + N(1 + \beta) + \mu)}{(1 + \gamma + \mu)} h$$

$$R^{k+1} = \frac{\mu S^k - \mu R^k - \mu I^k}{1 + \gamma + \mu} h$$

Thus $S^{k+1} \geq 0, I^{k+1} \geq 0, R^{k+1} \geq 0$, whenever the discrete variables are non-negative at the previous iteration. It remains to prove the positive invariance of $\kappa$.

Adding equation 4 & 5 we get

$$H^{k+1}(1 + \beta h) = \varnothing \mu N + H^k - (\varnothing(\gamma + \mu) + 1) I^{k+1} \leq \varnothing \mu N + H^k$$

$$H^{k+1}(1 + \beta h) \leq \varnothing \mu N + H^k$$

Therefore

$$H^{k+1} \leq N, \text{ whenever } H^k \leq N$$

The priori bounds for $R^{k+1}$ follow readily from the fact that $R^{k+1}$ and $I^{k+1}$ are less than or equal to $H^{k+1}$. This complete the proof.

**Numerical Simulations**

The mathematical analysis of epidemic measles SIR model with non-linear incidence has been presented. To observe the effects of the parameters using in this dynamics measles SIR model (1)-(3), conclude several numerical simulations with parameters values $\alpha = 0.003, \mu = 0.05, \gamma = 1, N = 1000$ are given in [16]. Hence $R_0 = 2.85$ and unique equilibrium point $E_1 = (350, 30.95, 619.04)$ is asymptotically stable, this result enhances Theorem 1. Figure 1-2 shows the convergence solution using NSFD scheme at $h = 10$ and $h = 1$ for true equilibrium point. It can be easily seen that by labeled points in figure 1-2, which shows fast convergence by decreasing the value of $h$. Figure 1-2 shows that the model presented gradually approaches the steady states for different values of $h$. In figure 1-2 clearly shows that infected individuals at latent period are diagnosed and treated, the number of susceptible individuals decreases significantly, the infected decreases steadily while the recovered increased steadily. The conclusion is that this paper deals with the mathematical model of measles disease to investigate the occurrence of diseases in population. The analysis of the system is well established. Qualitative analysis and stability analysis of the model according to equilibrium point are discussed and where it is locally asymptotically stable for $R_0 > 1$. The nonstandard finite difference scheme is dynamically consistent,
easy to implement and show a good agreement to control the spread of measles disease for long period of time. Finally we presented the numerical simulation and verified all the analytical results numerically by using nonstandard finite difference scheme to reduce infected rate for endemic equilibrium. While this conclusion may have practical implications for the control of measles infections, more realistic models that are specific for measles infection.

References